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Uncertainty Relations

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Abstract. The mutual relationship of the Heisenberg uncertainty relations, two recently derived uncertainty relations, the Robertson–Schrödinger uncertainty relation and the inequality for the Fisher information is discussed.

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HEISENBERG UNCERTAINTY RELATIONS

Similarly to [1, 2, 3] we write the wave function ψ in the form

$$\psi = e^{(is_1 - s_2)/\hbar}, \quad (1)$$

where $s_1 = s_1(x, t)$ and $s_2 = s_2(x, t)$ are real functions of the coordinate x and time t . The Heisenberg uncertainty relation for the coordinate x and momentum p has the form

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4}, \quad (2)$$

where

$$\langle(\Delta x)^2\rangle = \int (x - \langle x \rangle)^2 |\psi|^2 dx, \quad \langle(\Delta p)^2\rangle = \int |(\hat{p} - \langle \hat{p} \rangle)\psi|^2 dx, \quad (3)$$

$\hat{p} = -i\hbar(\partial/\partial x)$, $\langle \rangle$ denotes the usual quantum-mechanical mean value and integration is carried out from minus infinity to plus infinity.

Using Eqs. (1)–(3) we get [1, 2, 3]

$$\langle(\Delta p)^2\rangle = \langle(\Delta p_1)^2\rangle + \langle(\Delta p_2)^2\rangle, \quad (4)$$

where

$$\langle(\Delta p_1)^2\rangle = \int \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right)^2 e^{-2s_2/\hbar} dx, \quad \langle(\Delta p_2)^2\rangle = \int \left(\frac{\partial s_2}{\partial x} \right)^2 e^{-2s_2/\hbar} dx. \quad (5)$$

We see that the mean square deviation of the momentum $\langle(\Delta p)^2\rangle$ can be split into two parts. The first part $\langle(\Delta p_1)^2\rangle$ can be interpreted within the statistical generalization of classical mechanics in which the classical momentum $p = \partial S/\partial x_{cl}$ where S is the classical action and x_{cl} is the classical coordinate is replaced by $\partial s_1/\partial x$ and the probability density $\rho = |\psi|^2 = \exp(-2s_2/\hbar)$ is introduced. The second part $\langle(\Delta p_2)^2\rangle$ is proportional to one of the most important quantities appearing in mathematical statistics — the Fisher information I (see e.g. [4, 5])

$$I = \int \frac{1}{\rho} \left(\frac{\partial \rho}{\partial x} \right)^2 dx = \frac{4}{\hbar^2} \int \left(\frac{\partial s_2}{\partial x} \right)^2 e^{-2s_2/\hbar} dx = \frac{4}{\hbar^2} \langle(\Delta p_2)^2\rangle. \quad (6)$$

By using the Schwarz inequality $(u, u)(v, v) \geq |(u, v)|^2$, where $(u, v) = \int_{-\infty}^{\infty} u^* v dx$, u and v are complex functions and the star denotes the complex conjugate, it is easy to derive the inequality known from mathematical statistics (see e.g. [3, 4, 5])

$$\int (x - a)^2 \rho dx I \geq 1, \quad (7)$$

where a is a real number.

We note that for $\langle(\Delta p_1)^2\rangle = 0$ the Heisenberg uncertainty relation (2) is equivalent to inequality (7) for the Fisher information with $a = \langle x \rangle$.

TWO NEW UNCERTAINTY RELATIONS

Now we show that the Heisenberg uncertainty relation can be replaced by two uncertainty relations for $\langle(\Delta p_1)^2\rangle$ and $\langle(\Delta p_2)^2\rangle$ (see also [1, 2, 3]).

First, we take

$$u = \Delta x \sqrt{\rho}, \quad v = \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right) \sqrt{\rho}. \quad (8)$$

Then, the Schwarz inequality yields the first uncertainty relation

$$\langle(\Delta x)^2\rangle \langle(\Delta p_1)^2\rangle \geq \left[\int \Delta x \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right) e^{-2s_2/\hbar} dx \right]^2. \quad (9)$$

The function $\partial s_1/\partial x$ in the last integral corresponds to the classical momentum $\partial S/\partial x_{cl}$ and this relation has the usual meaning known from mathematical statistics [3]. Depending on the functions s_1 and s_2 , the square of the covariance of the coordinate and momentum at the right-hand side of this relation can have arbitrary values greater than or equal to zero.

The second uncertainty relation can be obtained in an analogous way for

$$u = \Delta x \sqrt{\rho}, \quad v = \left(\frac{\partial s_2}{\partial x} - \left\langle \frac{\partial s_2}{\partial x} \right\rangle \right) \sqrt{\rho} \quad (10)$$

with the result

$$\langle(\Delta x)^2\rangle \langle(\Delta p_2)^2\rangle \geq \left[\int (x - \langle x \rangle) \left(\frac{\partial s_2}{\partial x} - \left\langle \frac{\partial s_2}{\partial x} \right\rangle \right) e^{-2s_2/\hbar} dx \right]^2. \quad (11)$$

The right-hand side of this relation can be simplified [1, 2, 3] and yields the second uncertainty relation

$$\langle(\Delta x)^2\rangle \langle(\Delta p_2)^2\rangle \geq \frac{\hbar^2}{4}. \quad (12)$$

This uncertainty relation follows from the Schwarz inequality in a similar way as the first one, however, the covariance (u, v) is in this case constant and equals $\hbar/2 > 0$ independently of the concrete form of the function s_2 . We note also that relation (12) is for $\langle x \rangle = a$ equivalent to relation (7) for the Fisher information.

We see that the Heisenberg uncertainty relation (2) can be replaced by two more detailed uncertainty relations (9) and (12). First uncertainty relation (9) can be understood as the standard statistical inequality between the coordinate x and momentum represented by the function $p = \partial s_1/\partial x$. Second uncertainty relation (12) can be understood as the standard statistical inequality, too. However, because of the specific form of the covariance (u, v) which equals $\hbar/2$ independently of s_2 , the left-hand side of this relation must be greater than or equal to $\hbar^2/4$.

We note that the sum of uncertainty relations (9) and (12) is equivalent to the Robertson-Schrödinger relation for the coordinate and momentum [3]. The Heisenberg uncertainty relation (2) can be obtained from the sum of the uncertainty relations by neglecting the first term on its right-hand side. Therefore, uncertainty relations (9) and (12) are stronger than the corresponding Heisenberg and Robertson-Schrödinger uncertainty relations.

In the following two sections, two examples of uncertainty relations (9) and (12) are given.

FREE PARTICLE

We assume that the wave function of a free particle is at time $t = 0$ described by the gaussian wave packet

$$\psi(x, 0) = \frac{1}{\sqrt{a\sqrt{\pi}}} e^{-x^2/(2a^2) + ikx} \quad (13)$$

with the energy

$$E = \frac{\hbar^2}{4ma^2} + \frac{\hbar^2 k^2}{2m}, \quad (14)$$

where $a > 0$ and k are real constants. By solving the time Schrödinger equation we get

$$\psi(x,t) = \frac{1}{\sqrt{a\sqrt{\pi}}} \frac{\sqrt{1 - \frac{i\hbar t}{ma^2}}}{\sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}} \exp \left\{ -\frac{(x - \frac{\hbar k}{m}t)^2}{2a^2 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]} + i \left[\frac{kx + \frac{\hbar t x^2}{2ma^4} - \frac{\hbar k^2 t}{2m}}{1 + \left(\frac{\hbar t}{ma^2}\right)^2} \right] \right\}. \quad (15)$$

The corresponding functions s_1 and s_2 equal

$$s_1(x,t) = \hbar k \frac{x + \frac{\hbar t x^2}{2ma^4} - \frac{\hbar k}{2m}t}{1 + \left(\frac{\hbar t}{ma^2}\right)^2} - \hbar \arctan \frac{\hbar t}{ma^2}, \quad (16)$$

$$s_2(x,t) = \frac{\hbar}{2} \left\{ \frac{(x - \frac{\hbar k}{m}t)^2}{a^2 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]} - \ln \frac{1}{a\sqrt{\pi} \sqrt{1 + \left(\frac{\hbar t}{ma^2}\right)^2}} \right\}. \quad (17)$$

As it could be anticipated, the mean momentum and the mean coordinate have the form

$$\langle \hat{p} \rangle = \left\langle \frac{\partial s_1}{\partial x} \right\rangle = \hbar k, \quad \langle x \rangle = \frac{\hbar k}{m}t. \quad (18)$$

The mean square deviations of the coordinate and momentum are given by the equations

$$\langle (\Delta x)^2 \rangle = \frac{a^2}{2} \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2 \right] \quad (19)$$

and

$$\langle (\Delta p_1)^2 \rangle = \frac{\hbar^4 t^2}{2m^2 a^6 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2 \right]}, \quad \langle (\Delta p_2)^2 \rangle = \frac{\hbar^2}{2a^2 \left[1 + \left(\frac{\hbar t}{ma^2}\right)^2 \right]}. \quad (20)$$

The left-hand side and the right-hand side of relation (9) have the same value

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_1)^2 \rangle = \left\langle \Delta x \left(\frac{\partial s_1}{\partial x} - \left\langle \frac{\partial s_1}{\partial x} \right\rangle \right) \right\rangle^2 = \frac{\hbar^4 t^2}{4m^2 a^4}. \quad (21)$$

Therefore, uncertainty relation (9) is fulfilled with the equality sign.

Calculating the left-hand side of uncertainty relation (12) we obtain

$$\langle (\Delta x)^2 \rangle \langle (\Delta p_2)^2 \rangle = \frac{\hbar^2}{4} \quad (22)$$

and see that uncertainty relation (12) is fulfilled with the equality sign, too.

LINEAR HARMONIC OSCILLATOR

We assume that the wave function of the linear harmonic oscillator in the coherent state is at time $t = 0$ described by the gaussian wave packet

$$\psi(x,0) = \left(\frac{m\omega}{\hbar\pi} \right)^{1/4} e^{-(\xi - \xi_0)^2/2}, \quad (23)$$

where

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x, \quad \xi_0 = \sqrt{\frac{m\omega}{\hbar}}x_0 \quad (24)$$

and x_0 is the center of the packet. The corresponding energy E equals

$$E = \frac{m\omega^2 x_0^2}{2} + \frac{\hbar\omega}{2}. \quad (25)$$

By solving the time Schrödinger equation we get

$$\psi(x, t) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-i\omega t/2} e^{i(m\omega/\hbar)[x_0^2 \cos(\omega t) - 2xx_0] \sin(\omega t)/2} e^{-(m\omega/\hbar)[x - x_0 \cos(\omega t)]^2/2}. \quad (26)$$

The corresponding functions s_1 and s_2 equal

$$s_1(x, t) = -\hbar\omega t/2 + (m\omega)[x_0^2 \cos(\omega t) - 2xx_0] \sin(\omega t)/2, \quad (27)$$

$$s_2(x, t) = \frac{\hbar}{4} (\ln \hbar + \ln \pi - \ln m - \ln \omega) + \frac{m\omega}{2} [x - x_0 \cos(\omega t)]^2. \quad (28)$$

The mean momentum and the mean coordinate have the same form as in classical mechanics

$$\langle \hat{p} \rangle = \left\langle \frac{\partial s_1}{\partial x} \right\rangle = -m\omega x_0 \sin(\omega t), \quad \langle x \rangle = x_0 \cos(\omega t). \quad (29)$$

The mean square deviations of the coordinate and momentum are given by the equations

$$\langle (\Delta x)^2 \rangle = \frac{\hbar}{2m\omega}, \quad \langle (\Delta p_1)^2 \rangle = 0, \quad \langle (\Delta p_2)^2 \rangle = \frac{\hbar m\omega}{2}. \quad (30)$$

It means that uncertainty relations (9) and (12) have the form

$$0 = 0, \quad \langle (\Delta x)^2 \rangle \langle (\Delta p_2)^2 \rangle = \frac{\hbar^2}{4}. \quad (31)$$

The equality sign in uncertainty relations (9) and (12) is obtained if the functions s_1 and s_2 are quadratic functions of x of the form $p(t)x^2 + q(t)x + r(t)$, where real coefficients $p(t)$, $q(t)$ and $r(t)$ can depend on time [3]. All functions s_1 and s_2 given by Eqs. (16), (17), (27) and (28) fulfill this condition.

It is worth to notice that this condition for relation (12) is independent of the form of the function s_1 . Therefore, the equality sign in this relation can be achieved for much larger class of the wave functions than in case of the Heisenberg or Robertson–Schrödinger uncertainty relations. It is interesting not only from theoretical but also from the experimental point of view.

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CONCLUSIONS

Heisenberg and Robertson–Schrödinger uncertainty relations known from quantum mechanics follow from two stronger uncertainty relations (9) and (12).

First relation (9) can be understood as the inequality for the product of variances of the deviation of the coordinate x and momentum represented by the function $p = \partial s_1 / \partial x$ from their mean values which must be greater than or equal to the square of the covariance of these quantities.

Second relation (12) is equivalent to the above mentioned inequality (7) for the Fisher information. It can be also understood as the inequality between the variances and covariances of the deviation of the coordinate x and the function $\partial s_2 / \partial x$ from their mean values. However, the corresponding covariance is constant and equals $\hbar/2$. The square of the covariance then yields the constant $\hbar^2/4$ appearing at the right-hand side of the Heisenberg uncertainty relation.

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